

# ON THE CONDENSATION PROPERTY OF THE LAMPLIGHTER GROUPS AND GROUPS OF INTERMEDIATE GROWTH

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**ABSTRACT.** The aim of this short note is to revisit some old results about groups of intermediate growth and groups of the lamplighter type and to show that the Lamplighter group  $L = \mathbb{Z}_2 \wr \mathbb{Z}$  is a condensation group and has a minimal presentation by generators and relators. The condensation property is achieved by showing that  $L$  belongs to a Cantor subset of the space  $\mathcal{M}_2$  of marked 2-generated groups consisting mostly of groups of intermediate growth.

## 1. INTRODUCTION

The modern development of group theory requires significant use of methods of geometry, topology, probability and measure theory, the theory of models etc. The space  $\mathcal{M}_k$  of marked  $k$ -generated groups, introduced in [Gri84] plays an important role in this development. It is a compact totally disconnected metrizable space and it is important to know which groups belong to its perfect kernel (or condensation part), which is homeomorphic to a Cantor set. Groups in the perfect kernel are called *condensation groups*. The aim of this note is to revisit some results of [Gri84] and to use them to show that the so called Lamplighter group  $L = \mathbb{Z}_2 \wr \mathbb{Z}$ , which is a popular object of study (see for example [GZ01, GK12]) belongs to a Cantor set and hence is a condensation group.

Let  $\Omega = \{0, 1, 2\}^{\mathbb{N}}$  be the set of all infinite sequence over  $\{0, 1, 2\}$  with the product (Tychonoff) topology.

**Main Theorem.** *There exists a subset  $\mathcal{L} = \{(L_\omega, T_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_2$  with the following properties:*

- a)  $\mathcal{L}$  is homeomorphic to  $\Omega$  (and hence is a Cantor set),
- b) If  $\omega \in \Omega$  is not eventually constant, then  $L_\omega$  has intermediate growth.
- c) If  $\omega \in \Omega$  is a constant sequence then  $L_\omega \cong L$ .
- d) All groups in  $\mathcal{L}$  are condensation groups.

A simple argument shows that a group possessing an infinite minimal presentation is a condensation group. Surprisingly, it was observed in [BCGS14] that there are finitely generated groups which do not have a minimal presentation. It follows from [Bau61] that groups of the form  $H \wr G$  where  $H$  and  $G$  are infinite and

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finitely generated are not finitely presented and it was observed in [Cor11] that such groups are condensation groups. It is probably well known (as indicated in [BCGS14]) that the standard presentation

$$L = \langle s, t \mid s^2, [s, s^{t^i}] \ i \geq 1 \rangle$$

is minimal. A proof of this fact using ideas of [Bau61] is presented for completeness. This provides an alternative proof of the fact that  $L$  is a condensation group.

An effective way to build large families of condensation groups is to construct closed subsets  $X \subset \mathcal{M}_k$ ,  $k \geq 2$  homeomorphic to a Cantor set. Such families were constructed in [Gri84, Gri85, Cha00, Nek07]. It will be interesting to produce such families based on new ideas.

## 2. PRELIMINARIES

For a topological space  $X$ , let  $X'$  denote its set of accumulation points. For any ordinal  $\alpha$  define the spaces  $X^{(\alpha)}$  inductively as follows:  $X^{(0)} = X$ ,  $X^{(\alpha+1)} = (X^{(\alpha)})'$  and  $X^{(\lambda)} = \bigcap_{\beta < \lambda} X^{(\beta)}$  if  $\lambda$  is a limit ordinal. If  $X$  is a Polish space, (i.e.,

a completely metrizable, separable space) for some countable ordinal  $\alpha_0$  we will have  $X^{(\alpha_0)} = X^{(\alpha)}$  for all  $\alpha \geq \alpha_0$  (see [Kec95, Theorem 6.1]). The least ordinal with this property is called the *Cantor-Bendixson rank* of  $X$  and will be denoted by  $rk_{CB}(X)$ . The set  $X^{(\alpha_0)}$  is called the *perfect kernel* (or condensation part) of  $X$  which will be denoted by  $\kappa(X)$ . Note that if nonempty,  $\kappa(X)$  is homeomorphic to a Cantor set and  $\kappa(X)$  is empty if and only if  $X$  is countable. Points in  $\kappa(X)$  are called condensation points and can be characterized as points for which every open neighborhood is uncountable (see [Kec95, I.6]).

Let  $\mathcal{M}_k$  denote the space marked groups consisting of pairs  $(G, S)$  where  $G$  is a group and  $S$  is an ordered set of (not necessarily distinct) set of  $k$  generators. Two marked groups  $(G, S)$  and  $(H, T)$  in  $\mathcal{M}_k$  are identified whenever the map  $s_i \mapsto t_i, i = 1, \dots, k$  extends to an isomorphism. Two points  $(G, S)$  and  $(H, T)$  are of distance  $\leq 2^{-N}$  if the Cayley graphs of  $(G, S)$  and  $(H, T)$  have isomorphic balls of radius  $N$ . This (ultra) metric makes  $\mathcal{M}_k$  into a compact, totally disconnected, separable space. It follows from the definition that a sequence  $(G_n, S_n) \in \mathcal{M}_k$  converges to  $(G, S) \in \mathcal{M}_k$ , if and only if, for every element  $w \in F_k$  (the free group of rank  $k$ ), there exists  $N = N_w \geq 0$ , such that the relation  $w = 1$  holds in  $G$  if and only if it holds in  $G_n$  for  $n \geq N$ .

An important problem of geometric group theory (raised in [Gri05]) is the identification of  $rk_{CB}(\mathcal{M}_k)$  for  $k \geq 2$ . It follows from [Cor11] that the lower bound  $rk_{CB}(\mathcal{M}_k) > \omega^\omega$ ,  $k \geq 2$  holds. By a classical result of B.H. Neumann [Neu37] there exists uncountably many non-isomorphic 2-generated groups. Therefore  $\kappa(\mathcal{M}_k)$  is a Cantor set for all  $k \geq 2$ . A finitely generated group  $G$  is called a *condensation group*, if for some generating set  $S$  of size  $k$  the pair  $(G, S)$  belongs to  $\kappa(\mathcal{M}_k)$ . It

follows that this property does not depend on the generating set (see [dCGP07, Lemma 1]).

In [Gri84] the second author constructed Cantor sets  $\mathcal{G} \subset \mathcal{M}_k$  consisting essentially of groups of intermediate growth. Clearly, groups belonging to these families lie in the condensation part of  $\mathcal{M}_k$ . In general, it is a challenging problem to identify which groups are in the condensation part. It is expected that every group of intermediate growth is a condensation group. In contrast, it is easy to observe that virtually nilpotent groups are not condensation. In [Cor11, BCGS14] condensation properties of metabelian groups were considered and it was proven that restricted wreath products  $H \wr G$  of two finitely generated infinite groups are condensation groups [Cor11, Proposition 8.1]. Also, by [Cha00] every non-elementary hyperbolic group is a condensation group.

Let us briefly recall the groups constructed in [Gri84]. Although the original definition is in terms of measure preserving transformations of the unit interval, we will give here a definition in terms of automorphisms of rooted trees. Let  $\Omega$  denote the set of all infinite sequences over the alphabet  $\{0, 1, 2\}$ . We identify  $\Omega$  with the product  $\{0, 1, 2\}^{\mathbb{N}}$  and endow it with the product topology. Let  $\tau : \Omega \rightarrow \Omega$  be the shift transformation, i.e.,  $\tau(\omega)_n = \omega_{n+1}$ . For each  $\omega \in \Omega$  we will define a subgroup  $G_\omega$  of  $\text{Aut}(\mathcal{T}_2)$ , where the latter denotes the automorphism group of the binary rooted tree  $\mathcal{T}_2$  whose vertices are identified with the set of finite sequences  $\{0, 1\}^*$ . Each group  $G_\omega$  is the subgroup generated by the four automorphisms denoted by  $a, b_\omega, c_\omega, d_\omega$  whose actions onto the tree is as follows:

For  $v \in \{0, 1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$\begin{aligned} b_\omega(0v) &= 0\beta(\omega_1)(v) & c_\omega(0v) &= 0\zeta(\omega_1)(v) & d_\omega(0v) &= 0\delta(\omega_1)(v) \\ b_\omega(1v) &= 1b_{\tau(\omega)}(v) & c_\omega(1v) &= 1c_{\tau(\omega)}(v) & d_\omega(1v) &= 1d_{\tau(\omega)}(v), \end{aligned}$$

where

$$\begin{aligned} \beta(0) &= a & \beta(1) &= a & \beta(2) &= e \\ \zeta(0) &= a & \zeta(1) &= e & \zeta(2) &= a \\ \delta(0) &= e & \delta(1) &= a & \delta(2) &= a \end{aligned}$$

and  $e$  denotes the identity.

Note that from the definition, the following relations are immediate:

$$a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = e$$

Denoting by  $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ , we obtain a subset  $\{(G_\omega, S_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$ . In [Gri84] it was observed that this subset is not closed. It was also shown in [Gri84] that modifying countably many groups in this family, one obtains a closed subset  $\mathcal{G} = \{(G_\omega, S_\omega) \mid \omega \in \Omega\}$  with the following properties:

**Theorem 1** ([Gri84]).

- 1)  $\mathcal{G}$  is homeomorphic to  $\Omega$  via the map  $\omega \mapsto (G_\omega, S_\omega)$ ,

- 2) If in  $\omega \in \Omega$  all symbols  $\{0, 1, 2\}$  appear infinitely often, then  $G_\omega$  is a 2-group,
- 3) For  $\omega \in \Omega$  which is not eventually constant (i.e., is not constant after some point),  $G_\omega$  has intermediate growth,
- 4) If  $\omega \in \Omega$  is eventually constant, then  $G_\omega$  is virtually metabelian of exponential growth.

### 3. PROOF OF THE MAIN THEOREM

We start with the following basic lemma:

**Lemma 1.** *Suppose that  $\{(G_n, S_n)\}$  is a sequence in  $\mathcal{M}_k$  converging to  $(G, S)$ . Let  $F_k$  be the free group of rank  $k$ , with basis  $\{x_1, \dots, x_k\}$  and let  $\pi : F_k \rightarrow G$ ,  $\pi_n : F_k \rightarrow G_n$  be the canonical maps. Given  $w_1, \dots, w_m \in F_k$ , let  $T = \{\pi(w_1), \dots, \pi(w_m)\}$ ,  $T_n = \{\pi_n(w_1), \dots, \pi_n(w_m)\}$  and  $H = \langle T \rangle \leq G$ ,  $H_n = \langle T_n \rangle \leq G_n$ . Then the sequence  $\{(H_n, T_n)\}$  converges to  $(H, T)$  in  $\mathcal{M}_m$ .*

*Proof.* Let  $F_m$  be the free group of rank  $m$  with basis  $\{y_1, \dots, y_m\}$  and let  $\gamma : F_m \rightarrow H$  and  $\gamma_n : F_m \rightarrow H_n$  be the canonical maps. Also, let  $p : F_m \rightarrow F_k$  be the group homomorphism defined by  $p(y_i) = w_i$ ,  $i = 1, \dots, m$ . Note that we have the following:

$$\gamma_n = \pi_n \circ p \quad \text{for every } n$$

and

$$\gamma = \pi \circ p$$

It follows that, given  $w \in F_m$ ,  $w = 1$  in  $H$  if and only if  $p(w) = 1$  in  $G$ . This shows that the sequence  $\{(H_n, T_n)\}$  converges to  $(H, T)$  in  $\mathcal{M}_m$ .  $\square$

The following is a description of the structure of the group  $G_{000\dots}$ .

**Theorem 2.** *The group  $G_{000\dots}$  is isomorphic to the group  $L \rtimes \mathbb{Z}_2$  where  $L = \mathbb{Z}_2 \wr \mathbb{Z}$  is the Lamplighter group given by presentation  $\langle s, t \mid s^2, [s, s^{t^i}], i \geq 1 \rangle$  and  $\mathbb{Z}_2$  acts on  $L$  by the automorphism*

$$\begin{aligned} s &\mapsto s^t \\ t &\mapsto t^{-1} \end{aligned}$$

*Proof.* Let us denote  $G_{000\dots}$  by  $G$  and denote its canonical generators by  $a, b, c, d$ . Let  $H$  be the subgroup of  $G$  generated by the elements  $b, c, d, b^a, c^a, d^a$ . There exists an embedding (see [Gri84])

$$\begin{aligned} \psi : H &\rightarrow G \times G \\ b &\mapsto (a, b) \\ c &\mapsto (a, c) \\ d &\mapsto (1, d) \\ b^a &\mapsto (b, a) \\ c^a &\mapsto (c, a) \\ d^a &\mapsto (d, 1) \end{aligned}$$

Let  $D = \langle\langle d \rangle\rangle$  be the normal closure of  $d$  in  $G$ . By induction on word length one can see that  $D$  is an abelian group (see [Gri84, Lemma 6.1]).

We claim that  $D = \langle d^g \mid g \in \langle a, b \rangle \rangle$ . Let us denote the right hand side by  $T$ . Clearly  $T$  is contained in  $D$ . It suffices to show  $T$  is normal. Since  $bcd = 1$  it is enough to show that  $(d^g)^c \in T$  for all  $g \in \langle a, b \rangle$ . By induction on  $k$  one can see that the following equality holds:

$$\psi(d^{(ab)^n}) = \begin{cases} (1, d^{(ab)^k}) & , n = 2k \\ (d^{(ab)^k a}, 1) & , n = 2k + 1 \end{cases}$$

We will show by induction on  $|g|$  that  $(d^g)^c = (d^g)^b$ . Suppose  $|g| = 1$ , the case  $g = b$  is obvious since  $bcd = 1$ . If  $g = a$ , we have  $\psi((d^a)^b) = \psi((d^a)^c) = (d^a, 1)$  and hence  $(d^g)^c = (d^g)^b$ . Now assume  $|g| > 1$ . Since  $d^b = d$  we can assume that  $g$  starts with  $a$ . There are two cases, either  $g = (ab)^n$  or  $g = (ab)^n a$  for some  $n$ . In the first case (using induction assumption)

$$\psi((d^{(ab)^n})^c) = \begin{cases} (1, (d^{(ab)^k})^c) = (1, (d^{(ab)^k})^b) & , n = 2k \\ ((d^{(ab)^k a})^a, 1) = (d^{(ab)^k}, 1) & , n = 2k + 1 \end{cases}$$

and in the second case

$$\psi((d^{(ab)^n a})^c) = \begin{cases} ((d^{(ab)^k})^a, 1) & , n = 2k \\ (1, (d^{(ab)^k a})^c) = (1, (d^{(ab)^k})^b) & , n = 2k + 1 \end{cases}$$

In any case  $\psi((d^g)^c) = \psi((d^g)^b)$ . This shows  $T$  is normal and hence  $D = T$ .

Now letting

$$t_n = \begin{cases} d^{(ab)^n} & n \geq 0 \\ d^{(ab)^{-n-1}a} & n < 0 \end{cases}$$

we see that  $T = \langle t_n \mid n \in \mathbb{Z} \rangle$ . Looking at  $\psi(t_n)$  we see that the  $t_n$  are mutually distinct, therefore  $T \cong \prod_{\mathbb{Z}} \mathbb{Z}_2$ .

Since  $\psi((ab)^2) = (ba, ab)$ , it follows that the element  $ab$  is of infinite order in  $G$ .

We will show that the subgroups  $D$  and  $\langle ab \rangle$  intersect trivially. Suppose not, then  $d^g = (ab)^n$  for some  $g \in \langle a, b \rangle$  and  $n \in \mathbb{Z}$ . Necessarily  $n$  has to be even since left hand side of  $d^g = (ab)^n$  has even number of  $a$ 's. If  $n = 2k$  then  $\psi((ab)^{2k}) = ((ba)^k, (ab)^k)$  whereas  $\psi(d^g) = (d^h, 1)$  or  $(1, d^h)$  for some element  $h \in G$ . It follows that  $(ab)^k = 1$  which is a contradiction since  $ab$  has infinite order.

Now the subgroup  $K = D \rtimes \langle ab \rangle$  is isomorphic to  $(\mathbb{Z}_2^\infty) \rtimes \mathbb{Z} \cong \mathbb{Z}_2 \wr \mathbb{Z}$  which is the Lamplighter group. This is true since we have

$$t_n^{ab} = t_{n+1}, \quad n \in \mathbb{Z}$$

and hence the generator  $\langle ab \rangle$  acts on  $D$  by shifting its generators.

Conjugating the generators of  $K$  by the generators of  $G$  we see that  $K$  is a normal subgroup. The quotient  $G/D$  is isomorphic to the infinite dihedral group  $D_\infty$  (see [Gri84, Lemma 6.1]) and maps onto the quotient  $G/K$ . The kernel of

this homomorphism contains the image of  $ab$  in  $G/D$ . From this It follows that  $K$  has index 2 in  $G$ . Hence we have  $G = K \rtimes \langle a \rangle \cong L \rtimes \mathbb{Z}_2$ . Identifying  $s = d, t = ab$  we see that conjugation by  $a$  gives the asserted automorphism of  $K$ .  $\square$

For  $\omega \in \Omega$  let  $L_\omega = \langle d_\omega, ab_\omega \rangle \leq G_\omega$ . By virtue of the relations  $a^2 = b_\omega^2 = c_\omega^2 = d_\omega^2 = b_\omega c_\omega d_\omega = 1$  we see that  $L_\omega$  is a normal subgroup of index 2 in  $G_\omega$  and hence share many properties with  $G_\omega$ . Let us denote by  $T_\omega = \{d_\omega, ab_\omega\}$  and  $\mathcal{L}_\omega = \{(L_\omega, T_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_2$ .

**Proof of the main Theorem:**

a) Consider the map  $\phi : \Omega \rightarrow \mathcal{L}$  given by  $\omega \mapsto (L_\omega, T_\omega)$ .  $\phi$  is continuous since, if  $w_n$  converges to  $w$ , then by Theorem 1  $(G_{w_n}, S_{w_n})$  converges to  $(G_w, S_w)$  and hence by Lemma 1  $(L_{w_n}, T_{w_n})$  converges to  $(L_w, T_w)$ . To see that  $\phi$  is injective: By [Gri84, Section 5], the following is true: Given  $\omega_1 \neq \omega_2$  in  $\Omega$ , there exists  $u \in F_4$  (depending on  $\omega_1$  and  $\omega_2$ ), such that  $u$  is trivial in  $G_{\omega_1}$  and nontrivial in  $G_{\omega_2}$  (this amounts to saying that the map  $a \mapsto a, b_{\omega_1} \mapsto b_{\omega_2}, c_{\omega_1} \mapsto c_{\omega_2}, d_{\omega_1} \mapsto d_{\omega_2}$  does not extend to an isomorphism from  $G_{\omega_1}$  to  $G_{\omega_2}$  i.e.,  $(G_{\omega_1}, S_{\omega_1})$  and  $(G_{\omega_2}, S_{\omega_2})$  are distinct points in  $\mathcal{M}_4$ ). One observes that such  $u$  is a 2 power and (since  $L_\omega$  has index 2 in  $G_\omega$ ) its image in  $G_\omega$  lies in  $L_\omega$ . Therefore the image of  $u$  in  $L_{\omega_1}$  is trivial but its image in  $L_{\omega_2}$  is nontrivial which implies that  $(L_{\omega_1}, T_{\omega_1})$  and  $(L_{\omega_2}, T_{\omega_2})$  are distinct. This shows that  $\phi$  is injective and by compactness we have that  $\phi$  is a homeomorphism.

b) This follows from Theorem 1 and the fact that  $L_\omega$  has finite index in  $G_\omega$ .

c) By Theorem 2  $L_{000\dots}$  is isomorphic to  $L$  and it is immediate from the definition of the groups that  $G_{000\dots} \cong G_{111\dots} \cong G_{222\dots}$  and  $L_{000\dots} \cong L_{111\dots} \cong L_{222\dots}$ .

d) This follows from part a).

As a corollary we obtain the following:

**Corollary 1.** *The Lamplighter group  $L$  is a condensation group.*

#### 4. MINIMAL PRESENTATIONS OF THE LAMPLIGHTER GROUPS

For a subset  $A \subset G$  of a group let  $\langle\langle A \rangle\rangle$  denote the normal subgroup generated by  $A$ . A presentation  $\langle X \mid R \rangle$  is called *minimal* if for every  $r \in R$  we have  $r \notin \langle\langle R \setminus \{r\} \rangle\rangle$ . The following is well known.

**Proposition 1.** *Let  $\langle X \mid r_1, r_2, \dots \rangle$  be an infinite minimal presentation where  $|X| = k$ . Then the marked group  $(G, X)$  lies in the condensation part of  $\mathcal{M}_k$ .*

*Proof.* It is enough to show that any open ball around  $(G, X)$  is uncountable. Let  $B = B((G, X), 2^{-N})$  be a ball of radius  $2^{-N}$  around  $(G, X)$ . A marked group  $(H, T) \in \mathcal{M}_k$  lies in  $B$  if and only if for all  $w \in F_k$  such that  $|w| \leq 2N + 1$ , we have  $w = 1$  in  $G \iff w = 1$  in  $H$ . Let  $A = \{w \in F_k \mid |w| \leq 2N + 1 \text{ and } w = 1 \text{ in } G\}$ . Choose  $M = M(N) \in \mathbb{N}$  large enough so that  $A \subset \langle\langle r_1, r_2, \dots, r_M \rangle\rangle$ . For any subset  $U \subset \mathbb{N}$  such that  $\{1, 2, \dots, M\} \subset U$ , let  $(G_U, X)$  be the group  $\langle X \mid r_i, i \in U \rangle$ . Clearly all  $(G_U, X) \in B$  and since the

initial presentation is minimal all of them are distinct marked groups. Hence  $B$  is uncountable.  $\square$

We will give an alternative proof of Corollary 1 by showing that the standard presentation of  $L$  is minimal.

For a group  $G$  and a subset  $S \subset G$  let

$$T_S = \{(s_1g, s_2g) \mid s_1, s_2 \in S, g \in G\} \subset G \times G$$

**Theorem 3.** [Bau61] *Let  $G$  and  $H$  be two groups and  $S \subset G$  be a subset. Then there exists a group  $W = W(H, G, S)$  (called the circle product of  $G$  and  $H$  with respect to  $S$ ) with the following properties:*

- $W$  contains subgroups  $H_g, g \in G$  all isomorphic to  $H$ ,
- $W$  is generated by  $G$  and  $H_1$ ,
- The subgroup  $K = \langle H_g \mid g \in G \rangle$  is normal in  $W$  and  $W = K \rtimes G$ ,
- For  $h_{g_1} \in H_{g_1}$  and  $g_2 \in G$  we have  $h_{g_1}^{g_2} \in H_{g_1g_2}$ ,
- $[H_{g_1}, H_{g_2}] = 1$  if and only if  $(g_1, g_2) \in T_S$ .

Note that  $W$  can also be realized by using graph products: Let  $\Gamma$  be the graph with vertex set  $G$  and edges  $T_S$ , and let  $K$  be the graph product where each vertex group is  $H$ . Clearly  $G$  acts on  $K$  and one can see that  $W \cong K \rtimes G$ .

**Proposition 2.** *For every  $n \geq 2$ , the presentation*

$$\langle s, t \mid s^n, [s, s^{t^i}] \ i \geq 1 \rangle$$

*is a minimal presentation of  $\mathbb{Z}_n \wr \mathbb{Z}$ .*

*Proof.* Clearly the relation  $s^n$  is not redundant. For  $i \geq 1$  let  $r_i = [s, s^{t^i}]$  and suppose that for some  $m \geq 1$   $r_m$  is redundant. Let

$$a_0 = 0 \quad , \quad a_{2j} = j(m+1) + (1 + \dots + j) \quad j \geq 1$$

and

$$a_{2j+1} = \begin{cases} a_{2j} + j + 1 & \text{if } j < m - 1 \\ a_{2j} + j + 2 & \text{if } j \geq m - 1 \end{cases} \quad , \quad j \geq 0 .$$

Note that

$$a_{2j+1} - a_{2j} = \begin{cases} j + 1 & \text{if } j < m - 1 \\ j + 2 & \text{if } j \geq m - 1 \end{cases} \quad , \quad j \geq 0$$

and

$$|a_k - a_\ell| > m \quad \text{if } |k - \ell| \geq 2.$$

Finally let  $S = \{a_0, a_1, a_2, \dots\} \subset \mathbb{Z}$  and observe that the set  $S - S = \mathbb{Z} \setminus \{-m, m\}$ . Form the circle product  $W = W(\mathbb{Z}_n, \mathbb{Z}, S)$  with generators  $x, y$ . By the properties of  $W$  we have for  $i \geq 1$

$$[x, x^{y^i}] = 1 \iff (0, i) \in T_S \iff i \in S - S \iff i \neq m.$$

Therefore, under the assumption that  $r_m$  is redundant in  $\mathbb{Z}_n \wr \mathbb{Z}$ , the map  $s \mapsto x, t \mapsto y$  defines a homomorphism from  $\mathbb{Z}_n \wr \mathbb{Z}$  to  $W$  which contradicts the fact that  $r_m = 1$  in  $\mathbb{Z}_n \wr \mathbb{Z}$  but  $[x, x^{y^m}] \neq 1$  in  $W$ .  $\square$

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